

IMC 2015, Blagoevgrad, Bulgaria

Day 2, July 30, 2015

Problem 6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

(Proposed by Ivan Krijan, University of Zagreb)

Solution. We prove that

$$\frac{1}{\sqrt{n}(n+1)} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}. \quad (1)$$

Multiplying by $\sqrt{n}(n+1)$, the inequality (1) is equivalent with

$$1 < 2(n+1) - 2\sqrt{n(n+1)}$$

$$2\sqrt{n(n+1)} < n + (n+1)$$

which is true by the AM-GM inequality.

Applying (1) to the terms in the left-hand side,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} \right) = 2.$$

Problem 7. Compute

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx.$$

(Proposed by Jan Šustek, University of Ostrava)

Solution 1. We prove that

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1.$$

For $A > 1$ the integrand is greater than 1, so

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} dx > \frac{1}{A} \int_1^A 1 dx = \frac{1}{A}(A-1) = 1 - \frac{1}{A}.$$

In order to find a tight upper bound, fix two real numbers, $\delta > 0$ and $K > 0$, and split the interval into three parts at the points $1 + \delta$ and $K \log A$. Notice that for sufficiently large A (i.e., for $A > A_0(\delta, K)$ with some $A_0(\delta, K) > 1$) we have $1 + \delta < K \log A < A$.) For $A > 1$ the integrand is decreasing, so we can estimate it by its value at the starting points of the intervals:

$$\begin{aligned} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx &= \frac{1}{A} \left(\int_1^{1+\delta} + \int_{1+\delta}^{K \log A} + \int_{K \log A}^A \right) < \\ &= \frac{1}{A} \left(\delta \cdot A + (K \log A - 1 - \delta) A^{\frac{1}{1+\delta}} + (A - K \log A) A^{\frac{1}{K \log A}} \right) < \end{aligned}$$

$$< \frac{1}{A} \left(\delta A + K A^{\frac{1}{1+\delta}} \log A + A \cdot A^{\frac{1}{K \log A}} \right) = \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Hence, for $A > A_0(\delta, K)$ we have

$$1 - \frac{1}{A} < \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx < \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Taking the limit $A \rightarrow \infty$ we obtain

$$1 \leq \liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \delta + e^{\frac{1}{K}}.$$

Now from $\delta \rightarrow +0$ and $K \rightarrow \infty$ we get

$$1 \leq \liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq 1,$$

so $\liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1$ and therefore

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1.$$

Solution 2. We will employ l'Hospital's rule.

Let $f(A, x) = A^{\frac{1}{x}}$, $g(A, x) = \frac{1}{x} A^{\frac{1}{x}}$, $F(A) = \int_1^A f(A, x) dx$ and $G(A) = \int_1^A g(A, x) dx$. Since $\frac{\partial}{\partial A} f$ and $\frac{\partial}{\partial A} g$ are continuous, the parametric integrals $F(A)$ and $G(A)$ are differentiable with respect to A , and

$$F'(A) = f(A, A) + \int_1^A \frac{\partial}{\partial A} f(A, x) dx = A^{\frac{1}{A}} + \int_1^A \frac{1}{x} A^{\frac{1}{x}-1} dx = A^{\frac{1}{A}} + \frac{1}{A} G(A),$$

and

$$\begin{aligned} G'(A) &= g(A, A) + \int_1^A \frac{\partial}{\partial A} g(A, x) dx = \frac{A^{\frac{1}{A}}}{A} + \int_1^A \frac{1}{x^2} A^{\frac{1}{x}-1} dx = \\ &= A^{\frac{1}{A}} + \left[\frac{-1}{\log A} A^{\frac{1}{x}-1} \right]_1^A = \frac{A^{\frac{1}{A}}}{A} - \frac{A^{\frac{1}{A}}}{A \log A} + \frac{1}{\log A}. \end{aligned}$$

Since $\lim_{A \rightarrow \infty} A^{\frac{1}{A}} = 1$, we can see that $\lim_{A \rightarrow \infty} G'(A) = 0$. Applying l'Hospital's rule to $\lim_{A \rightarrow \infty} \frac{G(A)}{A}$ we get

$$\lim_{A \rightarrow \infty} \frac{G(A)}{A} = \lim_{A \rightarrow \infty} \frac{G'(A)}{1} = 0,$$

so

$$\lim_{A \rightarrow \infty} F'(A) = \lim_{A \rightarrow \infty} \left(A^{\frac{1}{A}} + \frac{G(A)}{A} \right) = 1 + 0 = 1.$$

Now applying l'Hospital's rule to $\lim_{A \rightarrow \infty} \frac{F(A)}{A}$ we get

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = \lim_{A \rightarrow \infty} \frac{F(A)}{A} = \lim_{A \rightarrow \infty} \frac{F'(A)}{1} = 1.$$

Problem 8. Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75} .

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. Let $n = 26$, then $3^{75} = (n + 1)^{n-1}$. We use the following well-known

Lemma. If $f(x)$ is a polynomial of degree at most n , then its $(n + 1)$ -st finite difference vanishes: $\Delta^{n+1}f(x) := \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} f(x+i) \equiv 0$.

Proof. If Δ is the operator which maps $f(x)$ to $f(x + 1) - f(x)$, then Δ^{n+1} is indeed $(n + 1)$ -st power of Δ and the claim follows from the observation that Δ decreases the power of a polynomial.

In other words, $f(x) = \sum_{i=1}^{n+1} (-1)^{i+1} \binom{n+1}{i} f(x+i)$. Applying this for $f(x) = (n - x)^n$, substituting $x = -1$ and denoting $i = j + 1$ we get

$$(n + 1)^n = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} (n-j)^n = (n+1) \sum_{j=0}^n \binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n.$$

The j -th summand $\binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n$ may be interpreted as follows: choose j letters, consider all $(n-j)^n$ words without those letters and sum up $\frac{(-1)^j}{j+1}$ over all those words. Now we change the order of summation, counting at first by words. For any fixed word W with k absent letters we get $\sum_{j=0}^k \binom{k}{j} \cdot \frac{(-1)^j}{j+1} = \frac{1}{k+1} \cdot \sum_{j=0}^k (-1)^j \cdot \binom{k+1}{j+1} = \frac{1}{k+1}$, since the alternating sum of binomial coefficients $\sum_{j=-1}^k (-1)^j \cdot \binom{k+1}{j+1}$ vanishes. That is, after changing order of summation we get exactly initial sum, and it equals $(n + 1)^{n-1}$.

Solution 2 (based on students' papers). Define by W the set of all 26-strings of Latin letters, and let W^* be the set of all 25-strings of the alphabet $\{A, B, \dots, Z, *\}$. For any string in W or W^* , denote by $\ell(w)$ the number of different Latin letters (excluding stars) in w . According the definition, the weight of any word $w \in W$ is $\frac{1}{27-\ell(w)}$.

Define a map $\varphi : W \rightarrow W^*$ in the following way. Suppose that $w \in W$ is an arbitrary 26-string and x is its first letter. Remove the leading x from w and replace all other occurrences of x by the character $*$. Let $\varphi(w) \in W^*$ be the resulting 25-string. Notice that the number of different Latin letters decreases by 1, so we have $\ell(\varphi(w)) = \ell(w) - 1$.

The map φ is not injective. For every $u \in W^*$, as u contains $\ell(u)$ different characters, there are $26 - \ell(u)$ different Latin letters that are not used in u , there are $26 - \ell(u)$ possible pre-images of u . The pre-images of u contain $\ell(u) + 1$ Latin letters, so their weights are equal to $\frac{1}{26-\ell(u)}$. Hence, the total weight of the pre-images of u is equal to 1.

Therefore, the total weight of the words in W is equal to the number of strings in W^* , which is $|W^*| = 27^{25}$.

Problem 9. An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^tA$ where A^t is the transpose of A . For each n , determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

(Proposed by Shachar Carmeli, Weizmann Institute of Science)

Solution.

Answer: The maximum dimension of such a space is $\frac{n(n+1)}{2}$.

The number $\frac{n(n+1)}{2}$ can be achieved, for example the symmetric matrices are obviously t-normal and they form a linear space with dimension $\frac{n(n+1)}{2}$. We shall show that this is the maximal possible dimension.

Let M_n denote the space of $n \times n$ complex matrices, let $S_n \subset M_n$ be the subspace of all symmetric matrices and let $A_n \subset M_n$ be the subspace of all anti-symmetric matrices, i.e. matrices A for which $A^t = -A$.

Let $V \subset M_n$ be a linear subspace consisting of t-normal matrices. We have to show that $\dim(V) \leq \dim(S_n)$. Let $\pi : V \rightarrow S_n$ denote the linear map $\pi(A) = A + A^t$. We have

$$\dim(V) = \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi))$$

so we have to prove that $\dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)) \leq \dim(S_n)$. Notice that $\text{Ker}(\pi) \subseteq A_n$.

We claim that for every $A \in \text{Ker}(\pi)$ and $B \in V$, $A\pi(B) = \pi(B)A$. In other words, $\text{Ker}(\pi)$ and $\text{Im}(\pi)$ commute. Indeed, if $A, B \in V$ and $A = -A^t$ then

$$\begin{aligned} (A+B)(A+B)^t &= (A+B)^t(A+B) \Leftrightarrow \\ \Leftrightarrow AA^t + AB^t + BA^t + BB^t &= A^tA + A^tB + B^tA + B^tB \Leftrightarrow \\ \Leftrightarrow AB^t - BA &= -AB + B^tA \Leftrightarrow A(B+B^t) = (B+B^t)A \Leftrightarrow \\ &\Leftrightarrow A\pi(B) = \pi(B)A. \end{aligned}$$

Our bound on the dimension on V follows from the following lemma:

Lemma. Let $X \subseteq S_n$ and $Y \subseteq A_n$ be linear subspaces such that every element of X commutes with every element of Y . Then

$$\dim(X) + \dim(Y) \leq \dim(S_n)$$

Proof. Without loss of generality we may assume $X = Z_{S_n}(Y) := \{x \in S_n : xy = yx \quad \forall y \in Y\}$. Define the bilinear map $B : S_n \times A_n \rightarrow \mathbb{C}$ by $B(x, y) = \text{tr}(d[x, y])$ where $[x, y] = xy - yx$ and $d = \text{diag}(1, \dots, n)$ is the matrix with diagonal elements $1, \dots, n$ and zeros off the diagonal. Clearly $B(X, Y) = \{0\}$. Furthermore, if $y \in Y$ satisfies that $B(x, y) = 0$ for all $x \in S_n$ then $\text{tr}(d[x, y]) = -\text{tr}([d, x], y) = 0$ for every $x \in S_n$.

We claim that $\{[d, x] : x \in S_n\} = A_n$. Let E_i^j denote the matrix with 1 in the entry (i, j) and 0 in all other entries. Then a direct computation shows that $[d, E_i^j] = (j - i)E_i^j$ and therefore $[d, E_i^j + E_j^i] = (j - i)(E_i^j - E_j^i)$ and the collection $\{(j - i)(E_i^j - E_j^i)\}_{1 \leq i < j \leq n}$ span A_n for $i \neq j$.

It follows that if $B(x, y) = 0$ for all $x \in S_n$ then $\text{tr}(yz) = 0$ for every $z \in A_n$. But then, taking $z = \bar{y}$, where \bar{y} is the entry-wise complex conjugate of y , we get $0 = \text{tr}(y\bar{y}) = -\text{tr}(y\bar{y}^t)$ which is the sum of squares of all the entries of y . This means that $y = 0$.

It follows that if $y_1, \dots, y_k \in Y$ are linearly independent then the equations

$$B(x, y_1) = 0, \quad \dots, \quad B(x, y_k) = 0$$

are linearly independent as linear equations in x , otherwise there are a_1, \dots, a_k such that $B(x, a_1y_1 + \dots + a_ky_k) = 0$ for every $x \in S_n$, a contradiction to the observation above. Since the solution of k linearly independent linear equations is of codimension k ,

$$\begin{aligned} \dim(\{x \in S_n : [x, y_i] = 0, \text{ for } i = 1, \dots, k\}) &\leq \\ &\leq \dim(x \in S_n : B(x, y_i) = 0 \text{ for } i = 1, \dots, k) = \dim(S_n) - k. \end{aligned}$$

The lemma follows by taking y_1, \dots, y_k to be a basis of Y .

Since $\text{Ker}(\pi)$ and $\text{Im}(\pi)$ commute, by the lemma we deduce that

$$\dim(V) = \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)) \leq \dim(S_n) = \frac{n(n+1)}{2}.$$

Problem 10. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$M = \max_{0 \leq x \leq 1} |p(x)|.$$

For every positive integer k , let

$$J_k = \int_0^1 (p(x))^{2k} dx.$$

Obviously $0 < J_k < M^{2k}$ is a rational number. If $(p(x))^{2k} = \sum_{i=0}^{2kn} a_{k,i} x^i$ then $J_k = \sum_{i=0}^{2kn} \frac{a_{k,i}}{i+1}$. Taking the least common denominator, we can see that $J_k \geq \frac{1}{\text{lcm}(1, 2, \dots, 2kn+1)}$.

An equivalent form of the prime number theorem is that $\psi(N) = \log \text{lcm}(1, 2, \dots, N) \sim N$ if $N \rightarrow \infty$. Therefore, for every $\varepsilon > 0$ and sufficiently large k we have

$$\text{lcm}(1, 2, \dots, 2kn+1) < e^{(1+\varepsilon)(2kn+1)}$$

and therefore

$$M^{2k} > J_k \geq \frac{1}{\text{lcm}(1, 2, \dots, 2kn+1)} > \frac{1}{e^{(1+\varepsilon)(2kn+1)}},$$

$$M > \frac{1}{e^{(1+\varepsilon)(n+\frac{1}{2k})}}.$$

Taking $k \rightarrow \infty$ and then $\varepsilon \rightarrow +0$ we get

$$M \geq \frac{1}{e^n}.$$

Since e is transcendental, equality is impossible.

Remark. The constant $\frac{1}{e} \approx 0.3679$ is not sharp. It is known that the best constant is between 0.4213 and 0.4232. (See I. E. Pritsker, The Gelfond–Schnirelman method in prime number theory, *Canad. J. Math.* 57 (2005), 1080–1101.)