IMC 2015, Blagoevgrad, Bulgaria

Day 1, July 29, 2015

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(10 points)

Problem 2. For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

(10 points)

Problem 3. Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$ is a rational number.

(10 points)

Problem 4. Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$

(10 points)

Problem 5. Let $n \geq 2$, let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \ldots, A_{n+1}$. Prove that $\angle A_iBA_j > 90^\circ$ holds for at least n pairs (i, j) with $1 \leq i < j \leq n+1$.

(10 points)